# M.Sc. (Final) Degree Examination August / September 2009

## Directorate of Correspondence Course

## Mathematics

## Paper : PM-10.05 : Complex Analysis

## (Freshers)

Time : 3 Hours | Max. Marks : 80

**Note:**

1) **Answer any FIVE questions.**
2) **All questions carry EQUAL marks.**

1. a) If sum and product of two complex numbers are both real, show that the numbers are either real or one is conjugate of the other.

   b) If $|a_i| < 1$, $\lambda_i \geq 0$ for $i = 1, 2, \ldots\ldots n$ and $\lambda_1 + \lambda_2 + \ldots\ldots + \lambda_n = 1$, show that $|\lambda_1 a_1 + \lambda_2 a_2 + \ldots\ldots + \lambda_n a_n| < 1$.

   c) State and prove necessary and sufficient condition for f(z) to be analytic.

   **(3+3+10)**

2. a) Find C-R equations in polar form.

   b) Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$ and then determine its harmonic conjugate and analytic function.

   c) Show that inside the circle of convergence, $\sum_{n=0}^{\infty} a_n z^n$ represents an analytic function f(z) which is infinitely differentiable.

   **(4+4+8)**

3. a) State and prove Abel's limit theorem.

   b) Show that when a circle is transformed into a circle under the map $W = {}^1/_z$, the centre of the original circle is never mapped onto the centre of the image circle.

   c) Show that cross ratio is preserved under bilinear transformation.

4. a) Find the bilinear transformation which maps the points $z = 1, i, -i$ respectively onto the points $W = i, 0, -i$. Find the image of the region $|z| \leq 1$ from the transformation.

   b) State and prove Cauchy's theorem for a rectangle.

   c) State and prove Cauchy's inequality.

   **(4+7+5)**

5. a) Prove that the function which is analytic in the whole plane and satisfy the inequality $|f(z)| < |z|$, for some 'm' and all sufficiently large $|z|$ reduces to a polynomial.

   b) State and prove Taylor's Theorem.

   c) State and prove Schwarz Lemma. **(5+6+5)**

6. a) State and prove the argument principle. Give the interpretation of the name argument.

   b) Let f(z) be analytic in $0 < |z - a| < \delta$ and has a Laurent's series expansion (as applicable to the annulus $r < |z - a| < R$, with $r = 0$ and $R = \delta$),

   $f(z) = \sum_{-\infty}^{\infty} a_k (z - a)^k$.

   Then show that

   (i) f(z) has a removable singularity at $z = a$ if and only if $a_k = 0$, $k < 0$ i.e., if and only if its singular part is zero.

   (ii) f(z) has a pole at $z = a$ of order m if and only if $a_{-m} \neq 0$ and $a_k = 0$ for $k < -m$.

   (iii) f(z) has a essential singularity at $z = a$ if and only if $a_m \neq 0$ for infinitely many negative integers m. **(6+10)**

7. a) State Rouche's Theorem and use it to show that the equation $e^z = az^n$ (if $a > e$) has n roots inside the circle $|z| = 1$.

   b) Using residue theorem, solve any two of the following:

   (i) $\int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2}$ $(0 < P < 1)$

   (ii) $\int_{-\infty}^{\infty} \frac{x^4}{x^6 + a^6} dx$ $(a > 0)$

   (iii) $\int_0^{\infty} \frac{\cos ax}{(x^2 + b^2)^2} dx$ $(a > 0, b > 0)$

   **(6+10)**

8. a) Derive Poison's Integral formula.

   b) Show that $\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$ **(10+6)**

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