



DPA – 510 (Math)

M.Sc. (Previous) Examination, July/August 2011
(Directorate of Distance Education)
MATHEMATICS
Paper : PM 10-01 : Algebra

Time : 3 Hours

Max. Marks : 70/80

- Instructions :** 1) *Students who have attended 30 Marks IA Scheme will have to answer for total of 70 Marks.*
- 2) *Students who have attended 20 Marks IA Scheme will have to answer for total of 80 Marks.*
- 3) *Answer **any FIVE** questions from Section – A. Each question carries 14 marks for both 70/80 Marks Scheme and Q.No. 9 in Section – B is compulsory for 80 Marks Scheme.*

SECTION – A

1. a) Define a permutation group. Show that every permutation can be expressed as a product of transpositions in infinitely many ways.
- b) State and prove Cauchy's theorem for abelian groups.
- c) Show that the group of prime power order has a non-trivial centre. **(5+4+5)**
2. a) State and prove Sylow's theorem.
- b) Show that no group of order 108 is simple.
- c) Define a solvable group. Show that every subgroup of a solvable group is solvable. **(5+4+5)**
3. a) If in a ring R , $x^2 = x$ for every $x \in R$, then show that R is commutative.
- b) Define an integral domain. Show that every finite integral domain is a field.
- c) Show that every integral domain can be imbedded in a field. **(4+5+5)**

P.T.O.



4. a) Define a maximal ideal. Let R be the ring of all real valued continuous functions defined on the closed interval $[0, 1]$ and $M = \{f(x) \in R : f(1/3) = 0\}$. Show that M is a maximal ideal of R .
- b) State and prove unique factorization theorem for Euclidean rings. (7+7)
5. a) If v_1, v_2, \dots, v_n is a basis of V over F and if w_1, w_2, \dots, w_m in V are linearly independent over F , then show that $m \leq n$. Hence show that any two bases of a finite dimensional vector space have the same number of elements.
- b) Define quotient space V/W . If V is a finite dimensional vector space, then show that $\dim W \leq \dim V$ and $\dim V/W = \dim V - \dim W$.
- c) If V is a finite dimensional and W , a subspace of V , then show that
- $$\hat{W} \cong \hat{V}/A(W) \text{ and } \dim A(W) = \dim V - \dim W. \quad (4+5+5)$$
6. a) Define an algebra. Show that $A_F(V)$ is an algebra.
- b) Define the matrix representation of a linear transformation $T \in A_F(V)$. If A, B are matrices of T relative to the bases $\{v_1, v_2, \dots, v_n\}$ and $\{w_1, w_2, \dots, w_n\}$ respectively of V , then show that there exists an invertible matrix G such that
- $$B = G^{-1} A G.$$
- c) Let $V = \{f(x) \in F[x] : \deg f(x) \leq 3\}$. Define $T : V \rightarrow V$ by $T(f(x)) = f'(x)$. Verify the formula $B = G^{-1} A G$ of (f) in the bases $\{1, x, x^2, x^3\}$ and $\{1, 1+x, 1+x^2, 1+x^3\}$. (5+5+4)
7. a) Define characteristic root and characteristic vector of $T \in A_F(V)$. If $\dim_F V = n$, then show that T can have at most n distinct characteristic roots in F .
- b) If $T \in A_F(V)$ has all its characteristic roots in F , then show that there is a basis of V in which the matrix of T is triangular.
- c) Prove that two nilpotent linear transformations are similar if and only if they have the same invariants. (4+5+5)



8. a) If L is an algebraic extension of K and if K is an algebraic extension of F , then show that L is an algebraic extension of F .
- b) Let R be the field of reals and Q , the field of rationals, such that $\sqrt{2}$ and $\sqrt{3}$ are algebraic over Q .
- i) Exhibit a polynomial of degree 4 over Q satisfied by $\sqrt{2} + \sqrt{3}$.
- ii) Show that $Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$.
- iii) What is the degree of $\sqrt{2} + \sqrt{3}$ over Q ?
- c) Show that a polynomial $f(x)$ in $F[x]$ has a multiple root if and only if $f(x)$ and $f'(x)$ have a non-trivial common factor. (5+5+4)

SECTION – B

9. a) Find all possible Jordan Canonical forms for 8×8 matrix whose minimal polynomial is $x^2(x-1)^3$.
- b) Prove that the normal transformation N is
- i) Hermitian \Leftrightarrow its characteristic roots are real.
- ii) Unitary \Leftrightarrow its characteristic roots are of absolute value 1. (4+6)